ON THE VOLUME OF SOME SOLIDS.

From *Journal de Mathemtiques.*

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A BODY is comprised between two parallel planes, and a section made in this body by a plane parallel to the bases, at a distance x from the lower base, has an area equal to $\quad a+\beta x+\gamma x^2+\omega x^3,$ (1)
a, β, γ, ω being any four constant values, that is to say independent of x.

To determine the volume of this body.

Cut the body by an infinity of equidistant planes parallel to the given planes. In designating by S_0, S_1, S_2 . . . S_n the sections formed, S_0 and S_n designating the planes of the bases, the altitude h would be divided by these planes into n equal parts.

By virtue of the principle of Cavalieri (any body comprised between two parallel planes can be considered as the sum of an infinity of prisms or of cylinders of infinitely small altitude) the volume of the body considered would be expressed by

$$V=\frac{h}{n}\Big(S_0+S_1+S_2\;\ldots+S_{n-1}\Big),\text{ or } V=\frac{h}{n}\,\Sigma S. \tag{2}$$

This fixed, evaluate the sections S_0, S_1, S_2 . . . . . If $x=0$, $S_0=a$,

$$x=\frac{h}{n},\ S_1=a+\beta\frac{h}{n}+\gamma\frac{h^2}{n^2}+\omega\frac{h^3}{n^3},\ x=\frac{2h}{n},\ S_2=a+\beta\frac{2h}{n}+\gamma\frac{4h^2}{n^2}+\omega\frac{8h^3}{n^3},$$

$$x=\frac{3h}{n},\qquad S_3=a+\beta\frac{3h}{n}+\gamma\frac{9h^2}{n^2}+\omega\frac{27h^3}{n^3},$$

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$$x=\frac{(n-1)h}{n},\ S_{n-1}=a+\beta\frac{(n-1)h}{n}+\gamma\frac{(n-1)^2h^2}{n^2}+\omega\frac{(n-1)^3h^3}{n^3};$$

thence $\Sigma S=na+\beta\frac{h}{n}[1+2+3\,..+(n-1)]+\gamma\frac{h^2}{n^2}[1^2+2^2+3^2\,..+(n-1)^2]$

$$+\omega\frac{h^3}{n^3}[1^3+2^3+3^3\ldots+(n-1)^3],=na+\beta\frac{h}{n}\left[\frac{n(n+1)}{2}\right]$$
$$+\gamma\frac{h^2}{n^2}\left[\frac{n(n+1)\ (2n+1)}{6}\right]+\omega\frac{h^3}{n^3}\left[\frac{n^2(n+1)^2}{4}\right].$$

Replacing in (2) ΣS by this value, we have

$$V=ah+\beta\frac{h^2}{2}\left(1-\frac{1}{n}\right)+\gamma\frac{h^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\omega\frac{h^4}{4}\left(1-\frac{1}{n}\right)^2.$$

When n increases indefinitely, we have

$$V=ah+\beta\,\tfrac{1}{2}h^2+\gamma\,\tfrac{1}{3}h^3+\omega\,\tfrac{1}{4}h^4. \tag{A}$$

REMARK.—If we designate by B, B_1 the upper and lower bases of a body, and by B_2 the section equidistant from the bases we have

$$x = 0,\ B = a,\ x = h,\ B_1 = a + \beta h + \gamma h^2 + \omega h^3,$$

$$x = \tfrac{1}{2}h,\ B_2 = a + \beta\tfrac{1}{2}h + \gamma\tfrac{1}{4}h^2 + \omega\tfrac{1}{8}h^3;\ \text{whence}$$

$$4B_2 = 4a + 2\beta h + \gamma h^2 + \tfrac{1}{2}\omega h^3,\ \text{and consequently}$$

$$B + B_1 + 4B_2 = 6a + 3\beta h + 2\gamma h^2 + \tfrac{3}{2}\omega h^3,$$

which multiplied by $\frac{1}{6}h$ produces formula (A). We then have

$$V = \tfrac{1}{6}h(B + B_1 + 4B_2),\ \text{a known formula.}$$

For the Sphere.—If in (1) we make $a = 0$, $\beta = 2\pi R$, $\gamma = -\pi$, $\omega = 0$, that function reduces to

$$2\pi Rx - \pi x^2 = \pi x(2R - x),$$

which represents the area of a circle made in a sphere, at the altitude x.

In this case $B = 0$, $B_1 = 0$, $B_2 = \pi R^2$, $h = 2R$; thence

$$V = \tfrac{1}{3}R.4\pi R^2 = \tfrac{4}{3}\pi R^3.$$

For the spherical segment.— Making $B = 0$, $B_1 = \pi h(2R - h)$, $B_2 = \pi.\frac{1}{2}h(2R - \frac{1}{2}h)$, we have

$$V = \pi.\tfrac{1}{6}h^2(2R - h + 4R - h) = \tfrac{1}{3}\pi h^2(3R - h).$$

For the cone.—If h is the altitude of a cone, R the radius of the base, the radius made at a height x is $R' = R - (R \div h)x$; whence

$$\pi R'^2 = \pi R^2 - \frac{2\pi R^2}{h}x + \frac{\pi R^2}{h^2}x^2,$$

the expression to which (1) reduces itself when $a = \pi R^2$, $\beta = -2\pi R^2 \div h$, $\gamma = \pi R^2 \div h^2$, $\omega = 0$. Thence the cone $= \frac{1}{6}h(\pi R^2 + \pi R^2) = \frac{1}{3}\pi R^2 h$.

In the same way we obtain for the volume of the frustum of a cone

$$V = \tfrac{1}{6}h[\pi R^2 + \pi R^2 + \pi(r + R)^2] = \tfrac{1}{3}\pi h(R^2 + r^2 + Rr).$$

NOTE ON BILINEAR TANGENTIAL COORDINATES.

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TANGENTIAL equations are usually written as homogeneous, and so analogous to trilinear equations; but interesting relations to the ordinary Cartesian coordinates are thus unnoticed. If the general equation of the first degree in two variables be divided through by its absolute term, and signs changed, it takes the form

$$px + qy - 1 = 0.$$

In the Cartesian system the two coefficients of this equation, p and q, are the reciprocals of the intercepts of its locus upon the two axes; and a recip-